

# ODE Maximum Principle at Infinity and Non-Compact Solutions of IMCF in Hyperbolic Space

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## Abstract

In this work we extend the ODE Maximum principle of Hamilton [14] to non-compact hypersurfaces using the Omari-Yau maximum principle at infinity [4, 22, 23, 26]. As an application of this result, we investigate Inverse Mean Curvature Flow (IMCF) of non-compact hypersurfaces in hyperbolic space. Specifically, we look at bounded graphs over horospheres in  $\mathbb{H}^{n+1}$  and show long time existence of the flow as well as asymptotic convergence to horospheres.

## 1. Introduction:

Non-compact maximum principles are important to the study of non-compact solutions of geometric evolution equations where standard maximum principles do not apply. Using a maximum principle which follows from Huisken's monotonicity formula, Ecker and Huisken [9] were able to show convergence under MCF to a translating soliton for graphs over planes in  $\mathbb{R}^{n+1}$ , satisfying certain initial growth conditions. Later, they developed further interior estimates for non-compact MCF [10] as well as a non-compact maximum principle that works for a fairly general class of evolution equations with time dependent metrics including Ricci Flow.

The non-compact maximum principles mentioned above follow the standard parabolic PDE techniques where evolution equations need to be controlled on the whole domain of definition in order for the maximum principle

to apply. In the literature on MCF and IMCF, though, there have been examples of cases where evolution equations cannot be controlled on the whole domain of definition but where the specific geometry around a max or min can be exploited to control the equation at these points. This is where an ODE maximum principle, such as Hamilton's maximum principle [14, 20], is most valuable and why the ODE maximum principle at infinity, a non-compact version of Hamilton's work, is important to the study of non-compact evolution equations (See Theorem (4) for an illustrative example of this phenomenon).

To illustrate the importance of the ODE maximum principle at infinity we will apply it to the geometric evolution of hypersurfaces  $\Sigma^n$  through a one parameter family of embeddings  $\varphi : \Sigma \times [0, T) \rightarrow \mathbb{H}^{n+1}$ ,  $\varphi$  satisfying inverse mean curvature flow

$$\begin{cases} \frac{\partial \varphi}{\partial t}(p, t) = \frac{\nu(p, t)}{H(p, t)} & \text{for } (p, t) \in \Sigma \times [0, T) \\ F(p, 0) = \Sigma_0 & \text{for } p \in \Sigma \end{cases} \quad (1)$$

where  $H$  is the mean curvature of  $\Sigma_t := \varphi_t(\Sigma)$  and  $\nu$  is a consistently chosen normal vector (we will be more specific later).

Global existence results for initial hypersurfaces in euclidean space were first obtained by Gerhardt [11] and Urbas [25]. They independently proved that any compact, mean-convex and star-shaped hypersurface will asymptotically approach a sphere and converge to a sphere after an appropriate rescaling under IMCF (as well as a whole family of inverse flows).

Since then there have been extensions of this theorem to Lorentzian manifolds [12], hyperbolic space [13, 6] as well as to rotationally symmetric spaces with non-positive radial curvature [24]. There has also been a great deal of work on weak solutions of IMCF including viscosity solutions [5], weak solutions through connection to the p-Laplacian [21] as well as the most famous formulation of weak variational solutions to IMCF by Huisken and Ilmanen [17] which were used to prove the Riemannian Penrose Inequality (time symmetric case).

The non-compact case of IMCF has seen almost no attention besides the specific examples given by Huisken and Ilmanen [16] and the recent papers on solitons of IMCF by Drugan, Lee and Wheeler [7], Drugan, Fong and Lee [8], and Castro and Lerma [3]. Besides these examples of special solutions there has been no work on showing convergence to a prototypical hypersurface for a class of initial data as has been done for compact IMCF for the sphere.

The present work changes this by applying the ODE maximum principle at infinity to the study of non-compact IMCF in Hyperbolic space and more precisely we prove the following theorem

**Theorem 1.** *Let  $\Sigma_t$  be a smooth solution of IMCF with initial hypersurface  $\Sigma_0$  satisfying the following bounds on the mean curvature and second fundamental form,  $0 < H_0 \leq H(x, 0) \leq H_1 < \infty$  and  $|A|(x, 0) \leq A_0 < \infty$ . We further assume that  $\Sigma_0$  can be represented as a graph of a bounded function with bounded gradient, over and uniformly bounded away from  $\mathbb{R}^n \times \{0\}$  in the upper half space model of hyperbolic space. Then the IMCF starting at  $\Sigma_0$  exists for all time  $t \in [0, \infty)$  and the solution asymptotically converges to a horosphere.*

In the second section we state and prove an ODE maximum principle at infinity which allows us to use the Omari-Yau maximum principle at infinity [4, 22, 23, 26] to extend the ODE maximum principle of Hamilton [14], [20] to the case of bounded (in space) functions defined on non-compact domains.

In third section we use the ODE maximum principle at infinity to prove Theorem (1). In this section we focus on showing the proofs of estimates where some differences from the standard methods appear but we avoid showing all of the details once the usefulness of the ODE maximum principle at infinity is demonstrated.

## 2. ODE Maximum Principle at Infinity

In this section we state and prove an ODE maximum principle that works for functions defined on non-compact domains and will be applied to study non-compact solutions of IMCF in Hyperbolic space in the next section. This is an extension of the work of Hamilton [14] which is described in detail in [20].

**Theorem 2.** *Assume for  $t \in [0, T)$  that  $g(t)$  is a family of Riemannian metrics defined on the manifold  $M^n$  so that the dependence on  $t$  is smooth. We also assume that  $g_t$  is a metric to which the Omori-Yau maximum principle at infinity applies for each  $t \in [0, T)$ .*

*Let  $u : M \times [0, T) \rightarrow \mathbb{R}$  be a smooth function which is bounded for each time  $t \in (0, T)$ , i.e.  $|u(x, t)| \leq C(t)$ , satisfying*

$$(\partial_t - H^{ij} \nabla_i^{g_t} \nabla_j^{g_t}) u = \langle X(x, u, \nabla^{g_t} u, t), \nabla^{g_t} u \rangle_{g_t} + F(u)$$

where  $|X| \leq C_1(t)$ ,  $F$  is a locally Lipschitz function on  $\mathbb{R}$  and  $H_{ij}$  is a symmetric, positive definite matrix so that  $|H| \leq C_0$ .

Setting  $u_{sup}(t) = \sup_{x \in M} u(x, t)$  we have that the function,  $u_{sup}(t)$  is locally Lipschitz and hence differentiable at almost every time  $t \in [0, T)$ . At every differentiable time we have that

$$\frac{du_{sup}(t)}{dt} = \lim_{k \rightarrow \infty} \frac{\partial u(x_k, t)}{\partial t} \quad \text{where } \{x_k\} \subset \mathbb{R}^n \text{ is any sequence s.t. } \lim_{k \rightarrow \infty} u(x_k, t) = \sup_{x \in \mathbb{R}^n} u(x, t)$$

If  $\varphi : [0, T') \rightarrow \mathbb{R}$  is a maximal solution of the ODE

$$\begin{cases} \varphi'(t) &= F(\varphi(t)) \\ \varphi(0) &= u_{sup}(0) \end{cases}$$

then we have that  $u(x, t) \leq \varphi(t)$  for  $(x, t) \in M \times [0, \min\{T, T'\})$ .

**Note:** We did not impose conditions that imply the Omori-Yau maximum principle since there are fairly general assumptions that may be useful depending on the application. What is important is that you can guarantee that some hypotheses that guarantee the Omori-Yau maximum principle are in place in order to apply Theorem (2). With respect to our application of Theorem (2) to IMCF in Hyperbolic space, we will ensure that the hypotheses of Theorem (3) are satisfied in order to apply Theorem (2).

Before we can prove this theorem we will need the following lemma.

**Lemma 1.** *Let  $u : M^n \times (0, T) \rightarrow \mathbb{R}$  be a bounded  $C^1$  function then  $u_{sup} : (0, T) \rightarrow \mathbb{R}$ , defined as  $u_{sup}(t) = \sup_{x \in M} u(x, t)$ , is a locally Lipschitz function in  $(0, T)$ . Also, at every differentiable time  $t \in (0, T)$  we have that*

$$\frac{du_{sup}(t)}{dt} = \lim_{k \rightarrow \infty} \frac{\partial u(x_k, t)}{\partial t} \quad \text{where } \{x_k\} \subset M \text{ is any sequence s.t. } \lim_{k \rightarrow \infty} u(x_k, t) = \sup_{x \in M} u(x, t)$$

**Note:** If  $u$  attains its max at some point  $x \in M$  then we can take the trivial sequence which is constantly equal to  $x$ .

**Note:** This is an extension of Hamilton's work [14], explained in [21], to non-compact manifolds where we allow sup and inf instead of just max and min on compact manifolds.

*Proof.* Fix a  $t \in (0, T)$  and then choose a  $\delta > 0$  so that  $[t - \delta, t + \delta] \subset (0, T)$ . Then choose an  $\epsilon$  so that  $0 < \epsilon < \delta$  and note that since  $u$  is bounded and  $C^1$  on  $M \times (0, T)$  we know that for every  $x \in M$ , there exists some Lipschitz constant  $K > 0$ , depending on  $t$  and  $\epsilon$ , so that  $u(x, t + \epsilon) - u(x, t) \leq K\epsilon$ .

Now for each  $\epsilon > 0$  we can find a sequence  $\{x_k^\epsilon\}$  so that  $u_{sup}(t + \epsilon) = \lim_{k \rightarrow \infty} u(x_k^\epsilon, t + \epsilon)$  and hence

$$\begin{aligned} u_{sup}(t + \epsilon) &= \lim_{k \rightarrow \infty} u(x_k^\epsilon, t + \epsilon) \leq \limsup_{k \rightarrow \infty} u(x_k^\epsilon, t) + K\epsilon \\ &\leq \lim_{k \rightarrow \infty} u(x_k^0, t) + K\epsilon = u_{sup}(t) + K\epsilon \end{aligned}$$

where the second inequality follows from the fact that  $u_{sup}(t) = \lim_{k \rightarrow \infty} u(x_k^0, t)$ . So we have found that  $u_{sup}(t + \epsilon) - u_{sup}(t) \leq K\epsilon$ . Repeating this argument for  $-\delta < \epsilon < 0$  we conclude that  $u_{sup}$  is a locally Lipschitz function on  $(0, T)$  and hence differentiable at almost every time  $t$ .

Let  $t \in (0, T)$  be a time where  $u_{sup}$  is differentiable and let  $\{x_k\}$  be a sequence so that  $\lim_{k \rightarrow \infty} u(x_k, t) = \sup_{x \in M} u(x, t)$ . Then by the Mean Value Theorem, for every  $0 < \epsilon < \delta$  we can choose a  $s_k \in (t, t + \epsilon)$  so that  $u(x_k, t + \epsilon) = u(x_k, t) + \epsilon \frac{\partial u(x_k, s_k)}{\partial t}$  and so

$$\begin{aligned} u_{sup}(t + \epsilon) &\geq \limsup_{k \rightarrow \infty} u(x_k, t + \epsilon) = \limsup_{k \rightarrow \infty} \left[ u(x_k, t) + \epsilon \frac{\partial u(x_k, s_k)}{\partial t} \right] \\ &= u_{sup}(t) + \epsilon \limsup_{k \rightarrow \infty} \frac{\partial u(x_k, s_k)}{\partial t} \end{aligned}$$

so then by rearranging we find

$$\frac{u_{sup}(t + \epsilon) - u_{sup}(t)}{\epsilon} \geq \limsup_{k \rightarrow \infty} \frac{\partial u(x_k, s_k)}{\partial t}$$

and so by letting  $\epsilon \rightarrow 0$  we find that  $\frac{du_{sup}(t)}{dt} \geq \limsup_{k \rightarrow \infty} \frac{\partial u(x_k, t)}{\partial t}$ .

Now if we repeat this argument for  $-\delta < -\epsilon < 0$  we will get the following

$$\frac{du_{sup}(t)}{dt} \leq \liminf_{k \rightarrow \infty} \frac{\partial u(x_k, t)}{\partial t}$$

Putting this all together we see that

$$\limsup_{k \rightarrow \infty} \frac{\partial u(x_k, t)}{\partial t} \leq \frac{du_{sup}(t)}{dt} \leq \liminf_{k \rightarrow \infty} \frac{\partial u(x_k, t)}{\partial t}$$

which tells us that  $\lim_{k \rightarrow \infty} \frac{\partial u(x_k, t)}{\partial t}$  must converge at a differentiable time of  $u_{sup}(t)$  and equal its derivative.

□

*Proof.* By the previous Lemma we know that  $u_{sup}(t)$  is locally Lipschitz and hence differentiable almost everywhere in  $[0, T)$ . If we let  $t \in [0, T)$  be a differentiable time and  $\{x_k\}$  a sequence so that  $\lim_{k \rightarrow \infty} u(x_k, t) = \sup_{x \in M} u(x, t)$ ,  $|\nabla u(x_k, t)| < \frac{1}{k}$  and  $\nabla_i \nabla_j u(x_k, t) < \frac{1}{k} g_{ij}$ , which is guaranteed by the maximum principle at infinity, then we find

$$\begin{aligned} \frac{du_{sup}}{dt}(t) &= \lim_{k \rightarrow \infty} \frac{\partial u}{\partial t}(x_k, t) \\ &\leq \limsup_{k \rightarrow \infty} (H^{ij} \nabla_i \nabla_j u(x_k, t) + \langle X(x_k, u, \nabla u, t), \nabla u(x_k, t) \rangle + F(u(x_k, t))) \\ &\leq \limsup_{k \rightarrow \infty} \left( \frac{nC_0}{k} + \frac{|X|}{k} + F(u(x_k, t)) \right) \\ &\leq F \left( \limsup_{k \rightarrow \infty} u(x_k, t) \right) = F(u_{sup}(t)) \end{aligned}$$

and so we have that, at a differentiable time  $t$

$$\frac{du_{sup}}{dt}(t) \leq F(u_{sup}(t))$$

At this point we follow the argument from [20]. Now let  $\varphi : [0, T') \rightarrow \mathbb{R}$  be as in the statement of the Theorem and for  $\epsilon > 0$  let  $\varphi_\epsilon : [0, T_\epsilon) \rightarrow \mathbb{R}$  be the maximal solution of the family of ODEs

$$\begin{cases} \varphi'_\epsilon(t) &= F(\varphi_\epsilon(t)) \\ \varphi_\epsilon(0) &= u_{sup}(0) + \epsilon \end{cases}$$

Now we break the argument into three cases depending on the sign of  $F(u_{sup}(0))$  in order to show that  $T' \leq T_0$ .

**Case 1:**  $F(u_{sup}(0)) > 0$

For  $\epsilon_1 \leq \epsilon_2$ , small enough, we have that  $\varphi_{\epsilon_1}(t) \leq \varphi_{\epsilon_2}(t)$  (by ODE comparison) and hence  $T_{\epsilon_2} \leq T_{\epsilon_1}$ . So  $T_\epsilon$  is an increasing sequence of times as  $\epsilon \rightarrow 0$  and so  $T_\epsilon \nearrow T'$  as  $\epsilon \rightarrow 0$  and by the upper semi-continuity of the existence time w.r.t. the initial condition we have that  $T' \leq T_0$ .

**Case 2:**  $F(u_{sup}(0)) < 0$

For  $\epsilon_1 \leq \epsilon_2$ , small enough, we have that  $\varphi_{\epsilon_1}(t) \geq \varphi_{\epsilon_2}(t)$  (by ODE comparison) and hence  $T_{\epsilon_2} \geq T_{\epsilon_1}$ . So  $T_\epsilon$  is a decreasing sequence of times as  $\epsilon \rightarrow 0$  and so  $T_\epsilon \searrow T'$  as  $\epsilon \rightarrow 0$  and by the upper semi-continuity of the existence time w.r.t. the initial condition we have that  $T' \leq T_0$ .

**Case 3:**  $F(u_{sup}(0)) = 0$

If for all  $\epsilon$  small enough we have that  $F(u_{sup}(0) + \epsilon) > 0$  or  $F(u_{sup}(0) + \epsilon) < 0$  then we are back in Case 1 or Case 2. If for all  $\epsilon$  small enough  $F(u_{sup}(0) + \epsilon) = 0$  then  $T' = T_0 = \infty$ .

So we have shown that  $T' \leq T_0$  and since  $F$  is Lipschitz on compact sets we can restrict ourselves to  $[0, T_\delta]$  for  $T_\delta < T'$  where we know that  $u$  and  $\varphi_\epsilon$  are bounded, for small enough  $\epsilon$ , and hence solutions to the above ODE have continuous dependence on the initial conditions (over compact time intervals). Hence using the fact that the family of functions  $\varphi_\epsilon$  is uniformly Lipschitz for small enough  $\epsilon$  we find that  $\varphi_\epsilon \rightarrow \varphi$  uniformly on  $[0, T_\delta]$  for any  $T_\delta < T'$  as  $\epsilon \rightarrow 0$ .

Now fix  $\epsilon > 0$  and for sake of contradiction assume that there is some positive time so that  $u_{sup}(t) > \varphi_\epsilon(t)$  and let  $\bar{t} > 0$  be the infimum of all such times which we know is  $\neq 0$  since  $u_{sup}(0) = \varphi_\epsilon(0) - \epsilon$ . So  $u_{sup}(\bar{t}) = \varphi_\epsilon(\bar{t})$  and hence we can let  $\Phi_\epsilon(t) = \varphi_\epsilon(t) - u_{sup}(t)$ . Then at differentiable times for  $u_{sup}(t)$  in the interval  $[0, \bar{t})$  we know that  $\Phi_\epsilon(t) > 0$  and

$$\Phi'_\epsilon(t) \geq F(\varphi_\epsilon(t)) - F(u_{sup}(t)) \geq -C_\epsilon(\varphi_\epsilon(t) - u_{sup}(t)) = -C_\epsilon\Phi_\epsilon(t)$$

where  $C_\epsilon$  is a local Lipschitz constant for  $F$  in the interval  $\{\varphi_\epsilon(s) : 0 \leq s \leq \bar{t}\}$  and this differential inequality hold for a.e.  $t \in [0, \bar{t}]$ .

Then by integrating this equation we find that  $\Phi_\epsilon(t) \geq \Phi_\epsilon(0)e^{-C_\epsilon t} = \epsilon e^{-C_\epsilon t}$  and so in particular  $\Phi_\epsilon(\bar{t}) \geq \epsilon e^{-C_\epsilon \bar{t}} > 0$  but that contradicts the fact that  $\Phi_\epsilon(\bar{t}) = 0$ .

So  $u_{sup}(t) \leq \varphi_\epsilon(t)$  for every  $t \in [0, T_\delta)$  and so if we let  $\epsilon \rightarrow 0$  then we have that  $u_{sup}(t) \leq \varphi(t)$  for every  $t \in [0, T_\delta)$ . Since  $\delta > 0$  was arbitrary, we have proven the desired result for  $[0, T')$ .  $\square$

### 3. Non-Compact Solutions to IMCF in Hyperbolic Space

In this section we apply the ODE maximum principle at infinity to the study of non-compact solutions of IMCF in  $\mathbb{H}^{n+1}$ . Our aim is to highlight the differences from the compact case of IMCF but we do not intend to include all of the standard details. Therefore, once the usefulness of the ODE maximum principle has been demonstrated and the different details that show up in this case are illustrated we will point to standard references to finish the proof of Theorem (1). For detailed computations of all the evolution equations used in this paper as well as a thorough treatment of short time existence, similar to what is done in [12], see my dissertation [1] .

It is convenient for us to use the upper half space model of  $\mathbb{H}^{n+1}$  which is defined on the space  $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$  with coordinates  $(x_1, \dots, x_n, y)$  and the following metric

$$\bar{g} = \frac{1}{y^2} (dx_1^2 + \dots + dx_n^2 + dy^2)$$

where we denote the coordinate basis vectors as  $\partial_{x_1}, \dots, \partial_{x_n}, \partial_y = \partial_{x_{n+1}}$ . In particular we will be looking at solutions which can be written as graphs over  $\mathbb{R}^n \times \{0\}$ , i.e. if  $y(x, t) : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$  then  $\varphi(x, t) = (x, y(x, t))$  and  $\Sigma_t = \varphi(\mathbb{R}^n \times \{t\})$ . Then we can define  $v = \sqrt{1 + |\nabla^0 y|^2}$  where  $\nabla^0$  denotes derivatives w.r.t. the flat metric on  $\mathbb{R}^{n+1}$ . It will also be useful to define  $w = \bar{g}(\nu, \eta) = \frac{1}{vy}$ , where  $\eta = -\partial_y$  and  $\nu$  is the downward pointing normal (the downward pointing normal makes IMCF forwards parabolic). We will use  $\nabla$  and  $\langle \cdot, \cdot \rangle$  for the connection and metric w.r.t.  $\Sigma_t$ .

We start our study of non-compact solutions to IMCF in  $\mathbb{H}^{n+1}$  by looking at a concrete example of the evolution of horospheres in  $\mathbb{H}^{n+1}$  and then we



show that horospheres act as barriers in  $\mathbb{H}^{n+1}$  for hypersurfaces satisfying the hypotheses of Theorem 1.

**Example:** Consider the horosphere  $y = y_0$  as a graph over  $\mathbb{R}^n \times \{0\}$ . Then  $y$  is just a function of time and  $H = n$  and so we find the ODE

$$\frac{dy}{dt} = \frac{-y}{n}$$

which has the solution  $y(t) = y_0 e^{-t/n}$ .

In order to show that the above family of examples acts as a barrier for bounded graphs as in Theorem (1) we will need to apply the following version of the Omori-Yau maximum principle which will also allow us to apply Theorem (2) in order to obtain important estimates throughout this section.

**Theorem 3.** [23] *Let  $(M, g)$  be a complete, non-compact, Riemannian manifold. If  $p \in M$  then define  $r(x) : M \rightarrow \mathbb{R}$  to be the distance from  $x$  to  $p$  and assume that the radial Ricci curvature satisfies the following bound*

$$Rc(\nabla r, \nabla r) \geq -C(r^2 + 1)$$

*for some  $C > 0$ . Then for every bounded above function  $u \in C^2(M)$  there is a sequence of points  $\{x_n\} \subset M$  so that*

$$u(x_n) > \sup_M u - \frac{1}{n} \quad |\nabla u|(x_n) < \frac{1}{n} \quad \Delta u(x_n) < \frac{1}{n}$$

In the rest of this section we will apply the ODE maximum principle at infinity many times which rests on the application of the Omori-Yau maximum principle to  $\Sigma_t$  which we will justify now. Under the assumptions of Theorem (1) short time existence tells us that the bounds in Theorem (1) will hold for at least a short time. Then since  $H$  is bounded above and below and  $|A|$  is bounded above we know that  $|Rc|$  is bounded for a short time

which implies that Theorem (3) applies to  $\Sigma_t$  for at least a short time. The following estimates will show that it applies for all time.

The following Theorem demonstrates that the above example acts as a barrier for a certain class of non-compact solutions of IMCF.

**Theorem 4.** *If  $0 < \inf_{\mathbb{R}^n} y(x, 0) = y_0$  and  $\sup_{\mathbb{R}^n} y(x, 0) \leq y_1$  and we assume that  $\Sigma_0$  is a hypersurface to which the hypotheses of Theorem 1 apply then we find that*

$$y_0 e^{-t/n} \leq y(x, t) \leq y_1 e^{-t/n}$$

*So horospheres act as barriers for bounded graphs over  $\mathbb{R}^n$ .*

*Proof.* Notice that by assumption the function  $y(x, t)$  is bounded above and below and hence we can use the result that  $y_{inf}(t) = \inf_{\mathbb{R}^n} y(x, t)$  is a well defined, locally Lipschitz function. Then by Theorem (3) there exists  $\{x_k\} \in \mathbb{R}^n$  a sequence so that  $\lim_{k \rightarrow \infty} y(x_k, t) = \inf_{\mathbb{R}^n} y(x, t)$  then we know by the maximum principle at infinity that

$$|\nabla^0 y(x_k, t)| < \frac{1}{k} \quad \nabla^0 \nabla^0 y(x_k, t) > -\frac{1}{k} \delta$$

and so if we use the expressions for  $H$  and  $w$  in terms of graphs we find

$$\begin{aligned} H &= \frac{n + y \tilde{\delta}^{ij} y_{ij}}{\sqrt{1 + |\nabla^0 y|^2}} \Rightarrow H(x_k, t) \geq \frac{n - k^{-1} y \tilde{\delta}^{ij} \delta_{ij}}{\sqrt{1 + \frac{1}{k^2}}} \Rightarrow \lim_{k \rightarrow \infty} H(x_k, t) \geq n \\ w &= \frac{1}{y \sqrt{1 + |\nabla^0 y|^2}} \Rightarrow w(x_k, t) = \frac{1}{y(x_k, t) \sqrt{1 + |\nabla^0 y(x_k, t)|^2}} \Rightarrow \lim_{k \rightarrow \infty} w(x_k, t) = \frac{1}{y_{inf}(t)} \end{aligned}$$

Now we can find the following ODE for  $y(x, t)$

$$\frac{\partial}{\partial t} \left( \frac{1}{y^2} \right) = \frac{\partial}{\partial t} \bar{g}(\partial_y, \partial_y) = \frac{2}{H} \bar{g}(\bar{\nabla}_{\bar{v}} \partial_y, \partial_y) = \frac{2}{H} \bar{g}(-\frac{\bar{v}}{y}, \partial_y) = \frac{2}{yH} \bar{g}(\bar{v}, \eta) = \frac{2w}{yH}$$

If we let  $t$  be a point of differentiability of the locally Lipschitz function  $y_{inf}(t)$  and  $\{x_k\}$  a sequence such that  $y(x_k, t) \rightarrow y_{inf}(t)$  we find that

$$\frac{dy_{inf}(t)}{dt} = \lim_{k \rightarrow \infty} \frac{\partial y}{\partial t}(x_k, t) = - \lim_{k \rightarrow \infty} \frac{y^2 w}{H} \geq -\frac{1}{n} y_{inf}(t)$$

and so by integrating, since  $y_{inf}(t)$  is absolutely continuous, we find

$$y_{inf}(t) \geq y_0 e^{-t/n}$$

Using a similar argument for  $y_{sup}(t) = \sup_{\mathbb{R}^n} y(x, t)$  we find the other important estimate.  $\square$

**Note:** Theorem (4) is a simple example where the evolution of  $y$  cannot be controlled everywhere but can be controlled at the sup or inf by exploiting the maximum principle at infinity.

Now we obtain  $C^1$  bounds on  $y$  through the support function  $w = \bar{g}(\nu, \eta)$  since  $w^{-1} = yv$  (we already have a  $C^0$  bound from Theorem (4)).

**Theorem 5.** *If we assume that  $\Sigma_0$  is a hypersurface to which the hypotheses of Theorem (1) apply then we find that*

$$(i) \ w(x, t) \geq w_{inf}(0) e^{t/n} \qquad (ii) \ v(x, t) \leq \frac{y_{sup}(0)}{y_{inf}(0)} v_{sup}(0)$$

*Proof.* From the evolution equation for  $w^{-1}$  we find

$$(\partial_t - \frac{1}{H^2} \Delta) w^{-1} \leq -\frac{1}{n} w^{-1}$$

where we have used that  $|A|^2 \geq H^2/n$ . Now we can deduce the following differential inequality (at points of differentiability of  $w_{sup}$  using Theorem (2))

$$\frac{dw_{sup}^{-1}}{dt} \leq -\frac{1}{n} w_{sup}^{-1}$$

from which the first estimate follows. Then if we notice that  $w^{-1} = vy$  we can find the second estimate by combining with the estimate for  $y$  given in Theorem (4).  $\square$

Now we get the required bounds on  $H$  which shows that the operator defining IMCF remains uniformly parabolic.

**Theorem 6.** *If we assume that  $\Sigma_0$  is a hypersurface to which the hypotheses of Theorem (1) apply then we find*

$$c_0 \sqrt{n^2 + C_0 e^{-2t/n}} \leq H(x, t) \leq \sqrt{C_0 e^{-2t/n} + n^2}$$

where  $C_0 = H_{sup}(0)^2 - n^2$  if  $H_{sup}(0) > n$  and  $c_0 = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)H_{sup}(0)}$  or

$$c_0 \leq H(x, t) \leq n$$

where  $H_{sup}(0) \leq n$  and  $c_0 = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)}$ .

*Proof.* We have the evolution equation for  $H$

$$(\partial_t - \frac{1}{H^2} \Delta)H = -2 \frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} + \frac{n}{H}$$

and by short time existence we know that  $H$  is bounded above for at least a short time  $t$  and so by using the ODE maximum principle at infinity (2) we obtain the differential inequality at points of differentiability of  $H_{sup}(t)$

$$\frac{dH_{sup}}{dt} \leq \frac{1}{nH_{sup}} (n^2 - H_{sup}^2)$$

from which it follows by integration that  $H_{sup}(t) \leq \sqrt{C_0 e^{-2t/n} + n^2}$  where  $C_0 = H_{sup}(0)^2 - n^2$  if  $H_{sup}(0) > n$  and  $C_0 = 0$  if  $H_{sup}(0) \leq n$ .

Now to obtain the lower bound on  $H$  we consider the evolution equation for  $u = \frac{1}{wH}$  given in [1, 18] and by using the ODE maximum principle at infinity we obtain the following differential inequality at points of differentiability of  $u_{sup}$

$$\frac{du_{sup}}{dt} = -\frac{nu_{sup}}{H^2} \leq -\frac{n}{n^2 + C_0 e^{-2t/n}} u_{sup}$$

which implies, by integrating, that  $u(x, t) \leq \frac{H_{sup}(0)u_{sup}(0)}{\sqrt{n^2 e^{2t/n} + C_0}}$  when  $H_{sup}(0) > n$  and then by using the definition of  $u = \frac{1}{Hw}$  and applying (4) we find

$$\begin{aligned} H &\geq \frac{w^{-1} \sqrt{n^2 e^{2t/n} + C_0}}{H_{sup}(0)u_{sup}(0)} = \frac{yv \sqrt{n^2 e^{2t/n} + C_0}}{H_{sup}(0)u_{sup}(0)} \\ &\geq \frac{y_{inf}(0)e^{-t/n} H_{inf}(0)w_{inf}(0) \sqrt{n^2 e^{2t/n} + C_0}}{H_{sup}(0)} = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)H_{sup}(0)} \sqrt{n^2 + C_0 e^{-2t/n}} \end{aligned}$$

which completes the lower estimate of  $H$  when  $H_{sup}(0) > n$ .

When  $H_{sup}(0) \leq n$  we get the simpler differential inequality at points of differentiability of  $u_{sup}$

$$\frac{du_{sup}}{dt} = -\frac{nu_{sup}}{H^2} \leq -\frac{u_{sup}}{n}$$

which implies, by integrating, that  $u(x, t) \leq u_{sup}(0)e^{-t/n}$  and then by using the definition of  $u = \frac{1}{Hw}$  and applying (4) we find

$$H \geq w^{-1}u_{sup}(0)e^{-t/n} = yv u_{sup}(0)^{-1}e^{t/n} \geq \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)}$$

which completes the lower estimate of  $H$  when  $H_{sup}(0) \leq n$ . □

To obtain an upper bound on  $|A|$ , the last estimate that we will show, we note that we cannot directly apply the ODE maximum principle at infinity to the maximum eigenvalue of  $A$ ,  $\lambda_{max}(x, t) = \max_{v \in T_x \Sigma_t, |v|=1} A(v, v)$  since this function is only locally Lipschitz and hence a laplacian does not exist, even almost everywhere. Since the proof of the ODE maximum principle at infinity relies on a comparison principle for the laplacian we will need to use another method which is used by Heidusch in [15], which will be discussed in the proof of the following estimate.

**Theorem 7.** *If we assume that  $\Sigma_0$  is a hypersurface to which the hypotheses of Theorem (1) apply then we find*

$$|A| \leq \left( \max_{t=0} w \right) \left( \max_{t=0} w^{-1} \right) \left( \max_{t=0} |A| \right) e^{\frac{(1+c_0^2)t}{c_0^2 n}}$$

where  $c_0 = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)}$ .

*Proof.* We will use a smooth approximation of the max function discussed in [15],  $\mu(A_i^j) = \tilde{\mu}(\lambda_1, \dots, \lambda_n)$  where  $\lambda_i$  are the eigenvalues of  $A_i^j$ , which is a symmetric function of the eigenvalues. Then Heidusch is able to derive the following useful evolution inequality where the extra term in the formula below comes from the fact that we are in Hyperbolic space

$$(\partial_t - \frac{1}{H^2} \Delta) \mu \leq \mu \frac{|A|^2}{H^2} + \frac{n}{H^2} \mu$$

and so combining with  $w^{-1}$  to get rid of the bad term we find

$$(\partial_t - \frac{1}{H^2} \Delta) (\mu w^{-1}) \leq \frac{n}{H^2} (\mu w^{-1}) \leq \frac{\mu w^{-1}}{c_0^2 n}$$

and now by the ODE maximum principle at infinity we deduce that

$$\begin{aligned} w^{-1} \mu &\leq \left( \max_{t=0} w^{-1} \mu \right) e^{\frac{t}{c_0^2 n}} \Rightarrow \mu \leq \left( \max_{t=0} w^{-1} \mu \right) w e^{\frac{t}{c_0^2 n}} \\ &\leq \left( \max_{t=0} w \right) \left( \max_{t=0} w^{-1} \right) \left( \max_{t=0} \mu \right) e^{\frac{(1+c_0^2)t}{c_0^2 n}} \end{aligned}$$

from which the desired result follows. □

Now we can prove a long time existence theorem as follows

**Theorem 8.** *Let  $\Sigma_0$  be a hypersurface satisfying the hypotheses of Theorem (1) then  $\Sigma_t$ , the corresponding solution to IMCF, exists for all time  $t \in [0, \infty)$ .*

The proof is exactly the same as [18] since we have upper and lower bounds on  $H$  as well as an upper bound on  $|A|$ .

Now we move on to discussing asymptotic analysis where our goal is to state precise theorems with brief proofs but the reader can refer to [1, 2, 11, 13, 24, 25] for further details. We start with a  $C^1$  asymptotic estimate.

**Lemma 2.** *For hypersurfaces satisfying the hypotheses of Theorem (1) the corresponding solution to IMCF in hyperbolic space satisfies*

$$v^2 - 1 = |\nabla^0 y|^2 \leq C e^{-2t/n}$$

*Proof.* If we define  $\psi = |\nabla^0 y|^2$  we can derive the following evolution equation [1, 2]

$$\frac{\partial \psi}{\partial t} \leq \frac{y^2}{H^2} \left( \tilde{\delta}^{ij} \nabla_i^0 \nabla_j^0 \psi + 2G^k \nabla_k^0 \psi + \frac{1}{v^2} \delta^{ij} \nabla_i^0 \psi \nabla_j^0 \psi - \frac{2n\psi}{y^2} \right)$$

where  $\tilde{\delta}^{ij} = \delta^{ij} - \frac{\nabla_i^0 y \nabla_j^0 y}{v^2}$ .

Now we can use this and Theorem (2) to derive a differential inequality for  $\psi_{sup}(t)$ , at points of differentiability

$$\frac{d\psi_{sup}}{dt} \leq \frac{-2n}{H^2} \psi_{sup} \leq \frac{-2n}{n^2 + C_0 e^{-2t/n}} \psi_{sup} \leq -2 \left( \frac{1}{n} - \bar{C} e^{-2t/n} \right) \psi_{sup}(t)$$

where we have used the bound  $H^2 \leq n^2 + C_0 e^{-2t/n}$  and chosen a constant  $\bar{C} > 0$ .

Now by integrating this differential inequality we find

$$\psi_{sup} \leq D e^{-2t/n - n e^{-2t/n}}$$

for some constant  $D > 0$  which implies that  $\psi = |\nabla^0 y|^2 = O(e^{-2t/n})$ , as desired.  $\square$

**Lemma 3.** *For hypersurfaces satisfying the hypotheses of Theorem (1) the corresponding solution to IMCF in hyperbolic space satisfies*

$$|\nabla^0 \nabla^0 y| \leq C$$

for  $(x, t) \in \mathbb{R}^n \times [0, \infty)$ .

*Proof.* Since we are considering hypersurfaces which can be written as graphs over  $\mathbb{R}^n$  we can rewrite IMCF in terms of  $y$  as the following PDE

$$\frac{\partial y}{\partial t} = \frac{-yv^2}{n + y\tilde{\delta}^{ij}y_{ij}} = F(x, y, \nabla^0 y, \nabla^0 \nabla^0 y)$$

where  $\tilde{\delta}^{ij} = \delta^{ij} - \frac{\nabla_i^0 y \nabla_j^0 y}{v^2}$ . We can check that is uniformly elliptic by calculating

$$\frac{\partial F}{\partial a_{kl}} = \frac{yv^2}{(n + y\tilde{\delta}^{ij}y_{ij})^2} y \tilde{\delta}^{kl} = \frac{y^2}{H^2} \tilde{\delta}^{kl}$$

and applying our previous estimates of  $y, H$  and  $v$ . In fact our operator satisfies the hypotheses of Theorem 6 of Chapter 5 of [19] which gives estimates on  $|\nabla^0 \nabla^0 y|$  in a bounded domain which just depend on bounds of the operator  $F$ .

Now by taking an exhaustion of  $\mathbb{R}^n$  by balls  $B(0, R)$  and considering the sets  $U_{R,T} = B(0, R) \times [0, T)$  we can apply Theorem 6 of Chapter 5 of [19] to find  $|\nabla^0 \nabla^0 y| \leq C_{R,T}$  for each set  $U_{R,T}$ . Since the bounds on  $F$  are uniform on  $\mathbb{R}^n \times [0, \infty)$  then they hold in the limit as  $R, T \rightarrow \infty$  to give us the uniform bound  $|\nabla^0 \nabla^0 y| \leq C$ , as desired.  $\square$

**Lemma 4.** *For hypersurfaces satisfying the hypotheses of Theorem (1) the corresponding solution to IMCF in hyperbolic space satisfies*

$$|A_{ij} - g_{ij}| \leq C e^{-(\gamma + \frac{1}{n})t}$$

where  $\gamma > 0$  and  $(x, t) \in \mathbb{R}^n \times [0, \infty)$ .

*Proof.* By a similar argument to Lemma (3) we can apply the  $C^{2,\alpha}$  estimates of [19] to find that  $\|y\|_{C^{2,\alpha}(U_{R,T})} \leq C$  where  $U_{R,T}$  is defined as in Lemma (3). Since this theorem of Krylov depends on estimates of the operator  $F$  and  $C^2$  bounds on  $y$ , both of which are uniform in  $R$  and  $T$ , we can take the limit as  $R, T \rightarrow \infty$  to find that  $\|y\|_{C^{2,\alpha}(\mathbb{R}^n \times [0, \infty))} \leq C$ . Now we can use a parabolic, multiplicative interpolation inequality

$$\|y\|_{C^2(\mathbb{R}^n \times [0, \infty))} \leq C \|y\|_{C^{2,\alpha}(\mathbb{R}^n \times [0, \infty))}^\tau \|y\|_{C^0(\mathbb{R}^n \times [0, \infty))}^{1-\tau} \leq C e^{-\gamma t}$$



where  $\tau + \gamma = 1$ .

Now we note that since  $\Sigma_t$  is a graph over  $\mathbb{R}^n$  we can write  $A_{ij} = \frac{1}{y^v} \nabla_i^0 \nabla_j^0 y + g_{ij}$  and so we find that  $|A_{ij} - g_{ij}| \leq C \frac{y}{v} |\nabla^0 \nabla^0 y| \leq C e^{(-\gamma t - \frac{1}{n} t)}$ , as desired.  $\square$

The last asymptotic estimate we would like is to improve Lemma (4) so that  $|A_{ij} - g_{ij}| \leq C e^{-2t/n}$ , which is the optimal decay rate we expect for IMCF in Hyperbolic space.

**Theorem 9.** *For hypersurfaces satisfying the hypotheses of Theorem (1) the corresponding solution to IMCF in hyperbolic space satisfies*

$$|A_{ij} - g_{ij}| \leq C e^{-2t/n}$$

*Proof.* If we define  $G = |A_{ij} - g_{ij}|^2 = |A|^2 - 2H + n$  then we can find the following evolution inequality for  $G$

$$\left( \partial_t - \frac{1}{H^2} \Delta \right) G \leq -\frac{4n}{H^2} G + C_1 e^{-(\frac{4}{n} + \gamma)t} + C_2 e^{-(\frac{3}{n} + 3\gamma)t} \quad (2)$$

which uses Lemma (4) and one should see [6] for more details. Using previous estimates from this paper we find

$$\left( \partial_t - \frac{1}{H^2} \Delta \right) G \leq -\frac{4}{n + C e^{-2t/n}} G + C_1 e^{-(\frac{4}{n} + \gamma)t} + C_2 e^{-(\frac{3}{n} + 3\gamma)t}$$

Now we can use an integrating factor to rewrite

$$\frac{d}{dt} \left( (1 + n e^{2t/n})^2 G_{sup} \right) \leq (1 + n e^{2t/n})^2 \left( C_1 e^{-(\frac{4}{n} + \gamma)t} + C_2 e^{-(\frac{3}{n} + 3\gamma)t} \right)$$

which implies, by integrating, that  $G_{sup}(t) \leq C e^{-\frac{4t}{n}} + C e^{-(\frac{3}{n} + 3\gamma)t}$ . Then by applying Theorem (2) we get the estimate  $G(x, t) \leq C e^{-\frac{4t}{n}} + C e^{-(\frac{3}{n} + 3\gamma)t}$ .

Now if  $3\gamma \geq \frac{1}{n}$  then we are done so if it is not,  $3\gamma < \frac{1}{n}$ , then we can recalculate the evolution inequality (2) with the new bound on  $G$  to find

$$\left(\partial_t - \frac{1}{H^2}\Delta\right) G \leq -\frac{4n}{H^2}G + C_1 e^{-(\frac{6}{n} + \frac{3\gamma}{2})t} + C_2 e^{-(\frac{9}{2n} + \frac{9}{2}\gamma)t} \quad (3)$$

Then using the same analysis as above we would find  $G(x, t) \leq C e^{-\frac{4t}{n}}$  since, when we integrate the right hand side of (3), all the terms will be negative and hence can be thrown out except for the constant which is then multiplied by the integrating factor yielding the correct asymptotic decay rate.

□

#### 4. Conclusion

In this paper we have seen the utility of the ODE maximum principle at infinity by using Theorem (2) to prove a new long time existence theorem and asymptotic analysis for non-compact solutions of IMCF in hyperbolic space, Theorem (1). We fully expect the ODE maximum principle at infinity to be useful to many more results in the study of non-compact solutions of any geometric evolution equation, especially when it is hard to control terms appearing in an evolution equation on the whole domain as in Theorem (4).

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